

SIMPLICIAL COMPLEXES OF WHISKER TYPE

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ABSTRACT. Let $I \subset K[x_1, \dots, x_n]$ be a zero-dimensional monomial ideal, and $\Delta(I)$ be the simplicial complex whose Stanley–Reisner ideal is the polarization of I . It follows from a result of Soleyman Jahan that $\Delta(I)$ is shellable. We give a new short proof of this fact by providing an explicit shelling. Moreover, we show that $\Delta(I)$ is even vertex decomposable. The ideal $L(I)$, which is defined to be the Stanley–Reisner ideal of the Alexander dual of $\Delta(I)$, has a linear resolution which is cellular and supported on a regular CW-complex. All powers of $L(I)$ have a linear resolution. We compute $\text{depth } L(I)^k$ and show that $\text{depth } L(I)^k = n$ for all $k \geq n$.

INTRODUCTION

Graphs with whiskers have first been considered by Villarreal in [20]. They all share the nice property that they are Cohen–Macaulay. Various extensions of this concept and generalizations of his result have been considered in the literature, see [2], [19], [14] and [8]. The edge ideal of a whisker graph is obtained as the polarization of a monomial ideal $I \subset S$, where $S = K[x_1, \dots, x_n]$ is the polynomial ring over a field K , I is generated in degree 2 and $\dim S/I = 0$. In particular, I contains the squares x_1^2, \dots, x_n^2 . More generally, given a simplicial complex Γ , the whisker complex $W(\Gamma)$ is studied in [11]. Its facet ideal is the polarization of a monomial ideal in S which contains all the x_i^2 . In [11] the Stanley–Reisner ideal of the Alexander dual of the independence complex of $W(\Gamma)$ is called the face ideal of a simplicial complex. Such face ideals, under a different name, have also been studied in [16]. In [11], as well as in [16], it is shown that face ideals have a linear resolution. In [16] Loiskekoski even shows that all powers of a face ideal have a linear resolution.

In the present paper we generalize the above mentioned results by considering the polarization of any monomial ideal $I \subset S$ with $\dim S/I = 0$. The simplicial complex $\Theta(I)$, whose facet ideal coincides with the polarization I^\wp of I , is called of *whisker type* – the whiskers being the simplices corresponding to the polarization of the pure powers contained in I . The independence complex of $\Theta(I)$, denoted $\Delta(I)$, is characterized by the property that the Stanley–Reisner ideal $I_{\Delta(I)}$ coincides with I^\wp . Note that $F \in \Delta(I)$ if and only if F does not contain any facet of $\Theta(I)$.

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Given an arbitrary monomial ideal $I \subset S$, a multicomplex is associated with I , as defined by Popescu and the second author in [12]. Soleyman Jahan defines in [18, Proposition 3.8] a bijection between the facets of the multicomplex given by I and the facets of the simplicial complex associated with I^\wp . In Theorem 1.1 we present a short proof of this bijection when $\dim S/I = 0$, by using multiplicity theory. This result allows us to describe in Corollary 1.2 the facets of $\Delta(I)$. By applying the Eagon–Reiner Theorem it is then shown in Corollary 1.3 that the ideal $L(I)$ has a linear resolution, where $L(I)$ is generated by the monomials $x_{1,a_1+1} \cdots x_{n,a_n+1}$ for which $x_1^{a_1} \cdots x_n^{a_n}$ is a monomial in S not belonging to I .

In the case that $\dim S/I = 0$, the case we consider here, the corresponding multicomplex is pretty clean, see [12]. Soleyman Jahan showed in [18, Theorem 4.3] that if I defines a pretty clean multicomplex, then the simplicial complex associated with I^\wp is clean, which, by a theorem of Dress [5], implies that the simplicial complex attached to I^\wp is shellable. Applied to our situation it follows that $\Delta(I)$ is shellable. We give a direct proof of this fact by showing that $L(I)$ has linear quotients. This provides an explicit shelling of $\Delta(I)$, and as a side result we obtain a formula for the Betti numbers of $L(I)$ in terms of the h -vector of S/I , see Corollary 2.2. We conclude Section 2 with Corollary 2.4, where it is shown that the minimal graded free resolution of $L(I)$ is cellular and supported on a regular CW-complex. The proof is based on a result of Dochtermann and Mohammadi [4, Theorem 3.10], who showed that the minimal graded free resolution of any ideal with regular decomposition function, as defined in [13], have such nice cellular structure.

In Section 3 we show that $\Delta(I)$ is not only shellable but even vertex decomposable. This was already known for whisker graphs (see [3, Theorem 4.4]). Finally in Section 4 we prove that all powers of $L(I)$ have linear quotients, see Theorem 4.1. Analyzing the linear quotients, the depth function $f(k) = \text{depth } S/L(I)^k$ can be computed. In Corollary 4.2 a formula for the depth function is given and $\lim_{k \rightarrow \infty} \text{depth } S/L(I)^k$ is determined.

1. THE FACETS OF THE INDEPENDENCE COMPLEX OF A WHISKER TYPE SIMPLICIAL COMPLEX

Throughout this paper S denotes the polynomial ring $K[x_1, \dots, x_n]$ and $I \subset S$ a monomial ideal with $\dim S/I = 0$, unless otherwise stated. The (finite) set of monomials in S which belong to S but not to I will be denoted by $\text{Mon}(S \setminus I)$. For an arbitrary monomial ideal I , we denote by $G(I)$ the unique minimal set of monomial generators of I . We will consider the polarization of I , denoted I^\wp . The polynomial ring in which I^\wp is defined will be denoted by S^\wp .

In the following theorem (cf. [18, Proposition 3.8]) we determine the set $\text{Min}(I^\wp)$ of minimal prime ideals of I^\wp .

Theorem 1.1. *Let $I \subset S$ be a monomial ideal with $\dim S/I = 0$. The map ϕ which assigns to each monomial $u = x_1^{a_1} \cdots x_n^{a_n} \in S \setminus I$ the monomial prime ideal $\phi(u) = (x_{1,a_1+1}, \dots, x_{n,a_n+1}) \subset S^\wp$, establishes a bijection between $\text{Mon}(S \setminus I)$ and $\text{Min}(I^\wp)$.*

Proof. We first observe that $\phi(\text{Mon}(S \setminus I)) \subset S^\wp$. Indeed, since $\dim S/I = 0$, there exists for each $1 \leq i \leq n$ an integer $b_i > 0$ such that $x_i^{b_i} \in I$ and $x_i^{b_i-1} \notin I$. It follows that S^\wp is the polynomial ring in the variables $x_{i,1}, \dots, x_{i,b_i}$ with $1 \leq i \leq n$. Now let $u = x_1^{a_1} \cdots x_n^{a_n} \in \text{Mon}(S \setminus I)$. Then $a_i < b_i$ for all i , and this implies that $\phi(u) \in S^\wp$.

Next we show that $\phi(\text{Mon}(S \setminus I)) \subset \text{Min}(I^\wp)$. In fact, let $u = x_1^{a_1} \cdots x_n^{a_n}$ be an element in $\text{Mon}(S \setminus I)$, and let $v \in G(I)$. We claim that there exists an integer i such that x_{i,a_i+1} divides v^\wp , where v^\wp is the polarization of v . From this claim it follows that $I^\wp \subset \phi(u)$. Since $\text{height } I^\wp = \text{height } I = n$ and since $\text{height } \phi(u) = n$, we then see that $\phi(u)$ is in fact a minimal prime ideal of I^\wp .

Let $v = x_1^{b_1} \cdots x_n^{b_n}$. In order to prove the claim, note that $v^\wp = \prod_{i=1}^n (\prod_{j=1}^{b_i} x_{i,j})$. Since v does not divide u , there exists an integer i such that $b_i > a_i$. Therefore, x_{i,a_i+1} divides v^\wp , as desired.

Clearly, ϕ is injective. We will show that $|\text{Mon}(S \setminus I)| = |\text{Min}(I^\wp)|$. This will then imply that $\phi: \text{Mon}(S \setminus I) \rightarrow \text{Min}(I^\wp)$ is bijective. In order to see that these two sets have the same cardinality we observe that the multiplicity $e(S/I)$ of S/I is equal to the length $\ell(S/I)$ of S/I , because $\dim S/I = 0$, see [1, Corollary 4.7.11(b)]. Since $\ell(S/I) = \dim_K S/I$ and since the elements of $\text{Mon}(S \setminus I)$ form a K -basis of S/I , we see that $e(S/I) = \dim_K S/I = |\text{Mon}(S \setminus I)|$. On the other hand, since S/I is isomorphic to S^\wp/I^\wp modulo a regular sequence of linear forms [9, Proposition 1.6.2], and since S^\wp/I^\wp is reduced and equidimensional, [1, Corollary 4.7.8] implies that $e(S/I) = e(S^\wp/I^\wp) = |\text{Min}(I^\wp)|$. \square

We denote by $\Delta(I)$ the simplicial complex whose Stanley-Reisner ideal is I^\wp . We view the variables $x_{i,j} \in S^\wp$ as the vertices of $\Delta(I)$. As an immediate consequence of Theorem 1.1 we obtain

Corollary 1.2. *Let \mathcal{S} be the set of variables of S^\wp . Then $F \subset \mathcal{S}$ is a facet of $\Delta(I)$ if and only if there exists $x_1^{a_1} \cdots x_n^{a_n} \in \text{Mon}(S \setminus I)$ such that*

$$F = \mathcal{S} \setminus \{x_{1,a_1+1}, \dots, x_{n,a_n+1}\}.$$

Since $\Delta(I)$ is Cohen–Macaulay, the Eagon–Reiner Theorem [6] (see also [9, Theorem 8.1.9]) implies that $I_{\Delta(I)^\vee}$ has a linear resolution. Here $\Delta(I)^\vee$ denotes the Alexander dual of $\Delta(I)$. Recall that, if Δ is an arbitrary simplicial complex on the vertex set $[n] = \{1, \dots, n\}$ and $I_\Delta = \bigcap_F P_F$ where $P_F = (x_i : i \in F)$, then I_{Δ^\vee} is generated by the monomials u_F where $u_F = \prod_{i \in F} x_i$. These facts applied to our case yield

Corollary 1.3. *The ideal $L(I)$ generated by the monomials $x_{1,a_1+1} \cdots x_{n,a_n+1}$, with $x_1^{a_1} \cdots x_n^{a_n} \in \text{Mon}(S \setminus I)$, has a linear resolution.*

In the following we consider the special case that $x_i^2 \in I$ for all i . In that case all other generators of I are square-free. In simplified notation, the polarization I^\wp of I is generated by the square-free monomials in I and by the monomials $x_i y_i$ for $i = 1, \dots, n$.

Let Γ be the simplicial complex with $I(\Gamma) = J$ and $W(\Gamma)$ be the simplicial complex with $I(W(\Gamma)) = (J, x_1 y_1, \dots, x_n y_n)$. The edges of $W(\Gamma)$ corresponding to the $x_i y_i$ are called the *whiskers* of $W(\Gamma)$ and $W(\Gamma)$ is called the *whisker complex* of Γ .

Given a simplicial complex Σ , the independence complex Λ of Σ is the simplicial complex such that $I_\Lambda = I(\Sigma)$. Notice that $F \in \Lambda$ if and only if no face of Σ is contained in F .

As a special case of Theorem 1.1 we recover [11, Theorem 1.1].

Corollary 1.4. *Let Γ be a simplicial complex on the vertex set $[n]$, $I' = I(\Gamma)$ the facet ideal of Γ and $W(\Gamma)$ its whisker complex. Let $I = (I', x_1^2, x_2^2, \dots, x_n^2)$. Then $\Delta(I)$ is the independence complex of $W(\Gamma)$ and $L(I)$ is generated by the monomials $\prod_{i \in [n] \setminus F} x_i \prod_{i \in F} y_i$ with $F \in \Delta$, where Δ is the independence complex of Γ .*

2. LINEAR QUOTIENTS

Let $I \subset S$ be a monomial ideal with $\dim S/I = 0$. The main purpose of this section is to show that $L(I)$ not only has a linear resolution, but even has linear quotients.

Theorem 2.1. *The ideal $L(I)$ has linear quotients.*

Proof. Let $u, v \in G(L(I))$, $u = x_{1,a_1+1} \cdots x_{n,a_n+1}$ and $v = x_{1,b_1+1} \cdots x_{n,b_n+1}$. We set $u \leq v$ if $a_i \leq b_i$ for all i , and extend this partial order to a total order on $G(L(I))$. We claim that, with respect to this total order of the monomial generators of $L(I)$, the ideal $L(I)$ has linear quotients. Indeed, let $x_{1,a_1+1} \cdots x_{n,a_n+1}$ be the largest element in $G(L(I))$. Then $u = x_1^{a_1} \cdots x_n^{a_n} \in \text{Mon}(S \setminus I)$ and $x_i u \in I$ for all i . Set $I' = I + (u)$. Then the polarization $(I')^\circ$ of I' is equal to $I_{\Delta(I')}$. Notice that $L(I') \subset L(I)$ and $\ell(S/I') < \ell(S/I)$. In particular, $L(I) = (L(I'), x_{1,a_1+1} \cdots x_{n,a_n+1})$. Arguing by induction on the length, we may assume that $L(I')$ has linear quotients. Thus we just need to compute the colon ideal $Q = L(I') : x_{1,a_1+1} \cdots x_{n,a_n+1}$. We claim that

$$(1) \quad Q = (x_{1,1}, x_{1,2}, \dots, x_{1,a_1}, x_{2,1}, \dots, x_{2,a_2}, \dots, x_{n,1}, \dots, x_{n,a_n}).$$

Suppose that $j \in \{1, \dots, a_i\}$ for some i . Then $x_1^{a_1} \cdots x_i^{j-1} \cdots x_n^{a_n} \in \text{Mon}(S \setminus I)$ and

$$\phi(x_1^{a_1} \cdots x_i^{j-1} \cdots x_n^{a_n}) = x_{1,a_1+1} \cdots x_{i,j} \cdots x_{n,a_n+1} \in L(I').$$

It follows that $x_{i,j} \in Q$.

On the other hand, the elements $v / \gcd(v, x_{1,a_1+1} \cdots x_{n,a_n+1})$ with $v \in G(L(I'))$ generate Q , see for example [9, Proposition 1.2.2]. In fact, let $v \in G(L(I'))$. Then $v = x_{1,c_1+1} \cdots x_{n,c_n+1}$ and $x_1^{c_1} \cdots x_n^{c_n} \in \text{Mon}(S \setminus I')$. There exists i such that $c_i < a_i$ because $x_i u \in I$ for all i . Hence x_{i,c_i+1} does not divide $x_{1,a_1+1} \cdots x_{n,a_n+1}$, and therefore x_{i,c_i+1} divides $v / \gcd(v, x_{1,a_1+1} \cdots x_{n,a_n+1})$. Since $c_i + 1 \leq a_i$, the desired conclusion follows. \square

Corollary 2.2. *For every $i \geq 0$,*

$$\beta_i(S^\circ / L(I)) = \sum_{j \geq 0} h_j \binom{j}{i-1},$$

where $h_j = h_j(S/I)$ is the j -th component of the h -vector of S/I . In particular, $\text{proj dim } S^\circ / L(I) = \max\{\deg u : u \in \text{Mon}(S \setminus I)\} + 1$.

Proof. As in the previous proof, let $u = x_1^{a_1} \cdots x_n^{a_n} \in \text{Mon}(S \setminus I)$ with $x_i u \in I$ for all i . Set $I' = I + (u)$, and consider the short exact sequence

$$0 \rightarrow L(I)/L(I') \rightarrow S^\wp/L(I') \rightarrow S^\wp/L(I) \rightarrow 0.$$

Notice that $L(I)/L(I') \cong S^\wp/Q(-n)$ with Q as in (1). Hence its minimal free resolution is the Koszul complex \mathbb{K} on the variables $x_{i,j}$ with $x_{i,j} \in G(Q)$. Thus the minimal free resolution of $S^\wp/L(I)$ can be obtained as a mapping cone of \mathbb{K} and the minimal free resolution of $S^\wp/L(I')$. Therefore $\beta_0(S^\wp/L(I)) = \beta_0(S^\wp/L(I'))$, and for $i \geq 1$ we obtain

$$\begin{aligned} \beta_i(S^\wp/L(I)) &= \beta_i(S^\wp/L(I')) + \text{rank}(K_{i-1}) = \beta_i(S^\wp/L(I')) + \binom{\deg u}{i-1} \\ &= \sum_{u \in \text{Mon}(S \setminus I)} \binom{\deg u}{i-1} = \sum_{j \geq 0} h_j \binom{j}{i-1}. \end{aligned}$$

□

It is easily seen that the geometric realization of $\Delta(I)$ is a sphere if I is a complete intersection, and a ball otherwise. Both topological spaces admit shellable triangulations, though in general not all triangulations of these spaces are shellable, see [17] and [15]. However, due to Theorem 2.1 we have

Corollary 2.3. *The simplicial complex $\Delta(I)$ is shellable.*

As a further consequence of Theorem 2.1 we have

Corollary 2.4. *The graded minimal free resolution of $L(I)$ is cellular and supported on a regular CW-complex.*

Proof. Since $L(I)$ has linear quotients we may apply [4, Theorem 3.10] and only need to show that $L(I)$ admits a regular decomposition function. In order to explain this, let $J = (u_1, \dots, u_m)$ be an ideal with linear quotients with respect to the given order of the generators. The *decomposition function* of J (with respect to the given order of the generators of J) is the map $b: \text{Mon}(J) \rightarrow G(J)$ with $b(u) = u_j$, where j is the smallest number such that $u \in (u_1, \dots, u_j)$. For each $u_j \in G(J)$, let $\text{set}(u_j)$ be the set of all x_i such that $x_i u_j \in (u_1, \dots, u_{j-1})$. According to [13], the decomposition function b is called *regular*, if $\text{set}(b(x_i u_j)) \subset \text{set}(u_j)$ for all $u_j \in G(J)$ and all $x_i \in \text{set}(u_j)$.

Now let $u \in G(L(I))$, $u = x_{1,a_1+1} \cdots x_{n,a_n+1}$. By (1) we have

$$\text{set}(u) = \{x_{1,1}, x_{1,2}, \dots, x_{1,a_1}, x_{2,1}, \dots, x_{2,a_2}, \dots, x_{n,1}, \dots, x_{n,a_n}\}.$$

Let $x_{i,j} \in \text{set}(u)$. Then $b(x_{i,j}u) = x_{i,j}(u/x_{i,a_i+1})$, and so

$$\text{set}(b(x_{i,j}u)) = \text{set}(u) \setminus \{x_{i,j+1}, \dots, x_{i,a_i}\} \subset \text{set}(u),$$

as desired. □

3. VERTEX DECOMPOSABILITY

In [3, Theorem 4.4] it was shown that for any graph, the independence complex of its whisker graph is vertex decomposable. Here we extend this result by showing that $\Delta(I)$ is vertex decomposable for any monomial ideal I with $\dim S/I = 0$. Recall that a simplicial complex Δ is called *vertex decomposable* if Δ is a simplex, or Δ contains a vertex v such that

- (i) any facet of $\text{del}_\Delta(v)$ is a facet of Δ , and
- (ii) both $\text{del}_\Delta(v)$ and $\text{link}_\Delta(v)$ are vertex decomposable.

Here $\text{link}_\Delta(v) = \{G \in \Delta : v \notin G \text{ and } G \cup \{v\} \in \Delta\}$ is the *link* of v in Δ and $\text{del}_\Delta(v) = \{G \in \Delta : v \notin G\}$ is the *deletion* of v from Δ .

A vertex v which satisfies condition (i) is called a *shedding vertex* of Δ .

For the proof of the next result we observe the following fact: let Δ be a simplicial complex and v a vertex not belonging to Δ . The *cone* of v over Δ , denoted by $v * \Delta$, is the simplicial complex such that $\mathcal{F}(v * \Delta) = \{\{v\} \cup F : F \in \mathcal{F}(\Delta)\}$. If Δ is vertex decomposable, then $v * \Delta$ is again vertex decomposable (with respect to the same shedding vertex).

Theorem 3.1. *Let I be a monomial ideal in $S = K[x_1, \dots, x_n]$ with $\dim S/I = 0$. Then $\Delta(I)$ is vertex decomposable.*

Proof. By assumption, for each $1 \leq i \leq n$ there exists $b_i \geq 1$ such that $x_i^{b_i} \in G(I)$. Then $\Delta(I)$ is a simplicial complex on $\mathcal{S} = \{x_{1,1}, \dots, x_{1,b_1}, \dots, x_{n,1}, \dots, x_{n,b_n}\}$. We proceed by induction on $\sum_{i=1}^n b_i$. If $\sum_{i=1}^n b_i = n$, then $I = (x_1, \dots, x_n)$, which is a trivial case. Suppose that $\sum_{i=1}^n b_i > n$. Hence we may assume $b_n > 1$.

We first show that the vertex $x_{n,1}$ is a shedding vertex of $\Delta(I)$. Clearly,

$$\text{del}_{\Delta(I)}(x_{n,1}) = \{F : F \in \Delta(I), x_{n,1} \notin F\} \cup \{F \setminus \{x_{n,1}\} : F \in \Delta(I), x_{n,1} \in F\}.$$

Obviously, any facet of $\text{del}_{\Delta(I)}(x_{n,1})$ with $x_{n,1} \notin F$ is a facet of $\Delta(I)$. On the other hand, if we consider $F \setminus \{x_{n,1}\}$ with $F \in \mathcal{F}(\Delta(I))$ and $x_{n,1} \in F$, then $F \setminus \{x_{n,1}\}$ is not a facet of $\text{del}_{\Delta(I)}(x_{n,1})$. Indeed, since $F \in \mathcal{F}(\Delta(I))$, there exists $u \in \text{Mon}(S \setminus I)$ such that $\phi(u) = P_{S \setminus F}$. Let t be the largest integer such that x_n^t divides u . Then $x_{n,t+1} \in P_{S \setminus F}$ and so $x_{n,j} \in F$ for all $j \neq t+1$. Since $x_{n,1} \in F$, we have $t+1 \neq 1$. Let $u' = u/x_n^t$. Then $u' \in \text{Mon}(S \setminus I)$ and $\phi(u') = P_{((S \setminus F) \setminus \{x_{n,t+1}\}) \cup \{x_{n,1}\}}$. Thus $G = (F \setminus \{x_{n,1}\}) \cup \{x_{n,t+1}\} \in \mathcal{F}(\Delta(I))$. Since $G \in \text{del}_{\Delta(I)}(x_{n,1})$, the claim follows. Consequently, $\mathcal{F}(\text{del}_{\Delta(I)}(x_{n,1})) = \{F : F \in \mathcal{F}(\Delta(I)), x_{n,1} \notin F\}$ which implies that $x_{n,1}$ is a shedding vertex of $\Delta(I)$.

We now prove that $\text{del}_{\Delta(I)}(x_{n,1})$ and $\text{link}_{\Delta(I)}(x_{n,1})$ are vertex decomposable.

First we consider $\text{del}_{\Delta(I)}(x_{n,1})$. Let J_1 be the ideal in S with $\text{Mon}(S \setminus J_1) = \{u : u \in \text{Mon}(S \setminus I), x_n \text{ does not divide } u\}$. Then $\Delta(J_1)$ is a simplicial complex on $\mathcal{S} \setminus \{x_{n,1}, \dots, x_{n,b_n}\}$. By using Corollary 1.2 we see that

$$\text{del}_{\Delta(I)}(x_{n,1}) = x_{n,b_n} * (x_{n,b_n-1} * (\dots * (x_{n,2} * \Delta(J_1)))).$$

Our induction hypothesis implies that $\Delta(J_1)$ is vertex decomposable, and hence $\text{del}_{\Delta(I)}(x_{n,1})$ is vertex decomposable.

As for $\text{link}_{\Delta(I)}(x_{n,1})$, let Γ be the simplicial complex whose faces are obtained from the faces of $\text{link}_{\Delta(I)}(x_{n,1})$ as follows: for every $F \in \text{link}_{\Delta(I)}(x_{n,1})$, we replace each $x_{n,j} \in F$ by $x_{n,j-1}$. Hence Γ is a simplicial complex on $\mathcal{S} \setminus \{x_{n,b_n}\}$ and $\Gamma \cong \text{link}_{\Delta(I)}(x_{n,1})$. Let J_2 be the monomial ideal in S such that $\text{Mon}(S \setminus J_2) = \{u/x_n : u \in \text{Mon}(S \setminus I), x_n \text{ divides } u\}$. Then Corollary 1.2 implies that $\Gamma = \Delta(J_2)$, which is vertex decomposable by induction hypothesis. It follows that $\text{link}_{\Delta(I)}(x_{n,1})$ is vertex decomposable, as desired. \square

4. POWERS

In this section we study the powers of $L(I)$. The main result is

Theorem 4.1. *Let $I \subset S$ be a monomial ideal with $\dim S/I = 0$. Then $L(I)^k$ has linear quotients for all k . In particular, all powers of $L(I)$ have a linear resolution.*

Proof. Any $u \in L(I)^k$ can be written in the form $u = u'_1 u'_2 \cdots u'_n$, where $u'_i = x_{i,j(i)_1} x_{i,j(i)_2} \cdots x_{i,j(i)_k}$ for $i = 1, \dots, n$ with $j(i)_1 \leq j(i)_2 \leq \cdots \leq j(i)_k$. We define a partial order on $G(L(I)^k)$ by setting $v \leq u$, if, with respect to the lexicographical order, $u'_i \leq v'_i$ for all i , and we extend this partial order to a total order on the set of monomial generators of $L(I)^k$.

Now let $v, u \in L(I)^k$ with $v < u$. We need to show that there exists $w \in L(I)^k$ with $w < u$ such that $w/\gcd(w, u)$ is of degree 1 and such that $w/\gcd(w, u)$ divides $v/\gcd(v, u)$. Indeed, since $v < u$, there exists i such that $u'_i < v'_i$ in the lexicographical order. Thus if $v'_i = x_{i,j'(i)_1} x_{i,j'(i)_2} \cdots x_{i,j'(i)_k}$ with $j'(i)_1 \leq j'(i)_2 \leq \cdots \leq j'(i)_k$, then there exists ℓ such that $j'(i)_s = j(i)_s$ for $s < \ell$ and $j'(i)_\ell < j(i)_\ell$. We let $w = w'_1 w'_2 \cdots w'_n$ with $w'_t = u'_t$ for $t \neq i$ and $w'_i = x_{i,j'(i)_\ell} (u'_i / x_{i,j(i)_\ell})$.

It is clear that $w < u$. Furthermore, $w \in G(L(I)^k)$. In fact, $u = u_1 \cdots u_k$ with $u_i \in L(I)$, and $x_{i,j(i)_\ell}$ divides one of these factors, say it divides u_r . Then $\bar{u}_r = x_{i,j'(i)_\ell} (u_r / x_{i,j(i)_\ell})$ belongs to $L(I)$ since $j'(i)_\ell < j(i)_\ell$, and hence $w = u_1 \cdots \bar{u}_r \cdots u_k$ belongs to $G(L(I)^k)$. Note further that $w/\gcd(w, u) = x_{i,j'(i)_\ell}$ and that $x_{i,j'(i)_\ell}$ divides $v/\gcd(v, u)$. This completes the proof. \square

Corollary 4.2. *For $i = 1, \dots, n$, let b_i be the smallest integer such that $x_i^{b_i} \in I$. Then*

$$\text{depth } S^\phi / L(I)^k = \sum_{i=1}^n b_i - \max\{\deg(\text{lcm}(u_1, \dots, u_k)) : u_1, \dots, u_k \in \text{Mon}(S \setminus I)\} - 1.$$

In particular, $\text{depth } S^\phi / L(I)^k = n - 1$ for all $k \geq n$, and

$$\text{depth } S^\phi / L(I)^k > \text{depth } S^\phi / L(I)^{k+1},$$

as long as $\text{depth } S^\phi / L(I)^k > n - 1$.

Proof. In general, let $J \subset K[x_1, \dots, x_n]$ be a graded ideal generated by a sequence f_1, \dots, f_s with linear quotients, and denote by $q_j(J)$ the minimal number of linear forms generating the ideal $(f_1, f_2, \dots, f_{j-1}) : f_j$. Then $\text{depth } K[x_1, \dots, x_n]/J = n - q(J) - 1$, where $q(J) = \max\{q_j(J) : 2 \leq j \leq s\}$, see [10, Formula (1)].

We apply this formula to $S^\wp/L(I)^k$. Since the Krull dimension of S^\wp is equal to $\sum_{i=1}^n b_i$, it remains to be shown that

$$(2) \ q(L(I)^k) = \max\{\deg(\text{lcm}(u_1, u_2, \dots, u_k)) : u_1, u_2, \dots, u_k \in \text{Mon}(S \setminus I)\}.$$

To see this, let $u_1, u_2, \dots, u_k \in \text{Mon}(S \setminus I)$ where $u_j = x_1^{a_1(j)} \cdots x_n^{a_n(j)}$ for $j = 1, \dots, k$. Then $u = u'_1 u'_2 \cdots u'_n$ with $u'_i = \prod_{j=1}^k x_{i, a_i(j)+1}$ is a generator of $L(I)^k$. We may assume that u is the j -th element in the given total order of the elements of $G(L(I)^k)$. As shown in the proof of Theorem 4.1, $q_j(L(I)^k)$ is the cardinality of the set

$$\{x_{1,1}, \dots, x_{1,c_1}, x_{2,1}, \dots, x_{2,c_2}, \dots, x_{n,1}, \dots, x_{n,c_n}\},$$

where $c_i = \max\{a_i(1), \dots, a_i(k)\}$ for $i = 1, \dots, n$. It follows that

$$q_j(L(I)^k) = \deg(\text{lcm}(u_1, u_2, \dots, u_k)),$$

and hence equation (2) follows.

Suppose now that $k \geq n$. Then we may choose $u_i = x_i^{b_i-1}$ for $i = 1, \dots, n$ and $u_i \in \text{Mon}(S \setminus I)$ arbitrary for $i > n$, and obtain $\deg(\text{lcm}(u_1, u_2, \dots, u_k)) = \sum_{i=1}^n (b_i - 1) = \sum_{i=1}^n b_i - n$. Since this is the largest possible least common multiple of sequences of elements of $\text{Mon}(S \setminus I)$, it follows that $\text{depth } S^\wp/L(I)^k = n - 1$ for all $k \geq n$.

Finally, suppose that $\text{depth } S^\wp/L(I)^k > n - 1$. Then the formula for $\text{depth } S^\wp/L(I)^k$ implies that $\max\{\deg(\text{lcm}(u_1, \dots, u_k)) : u_1, \dots, u_k \in \text{Mon}(S \setminus I)\} < \sum_{i=1}^n (b_i - 1)$.

Let $x_1^{a_1} \cdots x_n^{a_n} = \text{lcm}(u_1, \dots, u_k)$ attain this maximal degree. Since $\sum_{i=1}^n a_i < \sum_{i=1}^n (b_i - 1)$, there exists an index i such that $a_i < b_i - 1$. Let $u_{k+1} = x_i^{b_i-1}$. Then $\deg(\text{lcm}(u_1, \dots, u_k, u_{k+1})) > \deg(\text{lcm}(u_1, \dots, u_k))$. Consequently, $\text{depth } S^\wp/L(I)^k > \text{depth } S^\wp/L(I)^{k+1}$, as desired. \square

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